

BOUCHARD-KLEMM-MARINO-PASQUETTI CONJECTURE FOR  $\mathbb{C}^3$ 

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ABSTRACT. In this paper, we give a proof of the Bouchard-Klemm-Marino-Pasquetti conjecture for a framed vertex, by using the symmetrized Cut-Join Equation developed in a previous paper.

## 1. INTRODUCTION

In their seminal paper [2], Bouchard, Klemm, Marino and Pasquetti propose a new approach to compute both the open and closed Gromov-Witten invariants of local Calabi-Yau manifolds, including the mirrors of toric varieties. The approach is based on the matrix models theory of Eynard and Orantin. To each toric Calabi-Yau three-fold, there is an algebraic curve  $\Sigma$ , living inside  $\mathbb{C}^* \times \mathbb{C}^*$ , associated to the toric diagram. The curve  $\Sigma$  is called the framed mirror curve, with genus equal to the genus of the toric diagram. Bouchard, Klemm, Marino and Pasquetti conjecture that the Gromov-Witten invariants of the toric Calabi-Yau three folds can be computed by applying the recursion of Eynard and Orantin to the framed mirror curve.

The first example of Bouchard-Klemm-Marino-Pasquetti theory is the non-compact toric three fold  $\mathbb{C}^3$ , the so-called framed vertex, which is the building block of three dimensional toric varieties. In this case, the framed mirror curve is the algebraic curve  $x = y^f(1 - y)$ . The theory of Eynard and Orantin produce a topological recursion relations, when plug in the information of the framed mirror curve. On the other hand, there is an existing topological recursion for  $\mathbb{C}^3$ , i.e, the cut-join equation proved in [8] and its symmetrized version [3]. In this paper, we will prove that the Symmetrized Cut-Join Equation obtained in [3] implies the topological recursion of Eynard and Orantin, thus prove the Bouchard-Klemm-Marino-Pasquetti conjecture in the  $\mathbb{C}^3$  case.

Our strategy is strongly motivated by the recent work of Eynard-Mulase-Safnuk on the Bouchard-Marino conjecture. The Bouchard-Marino conjecture on Hurwitz numbers is a topological recursion of Eynard-Orantin type. This conjecture was first proved by Borot, Eynard, Mulase and Safnuk, using very deep results in matrix model theory. Later, Eynard, Mulase and Safnuk give another proof by comparing the conjectural recursion with the symmetrized cut-join equation of Hurwitz numbers, which was discovered by Goulden-Jackson-Vainshtein [5]. We now describe their second approach. First, writing the residues in the conjectural recursion as a contour integral, and a residue theorem calculus switch the calculation to the two nearby simple poles. This computation provide an equivalent form of the conjectural formula, which is an identity of polynomials. Then, by pushing forward the Symmetrized Cut-Join Equation of Goulden-Jackson-Vainstein via the projection  $\pi : \Sigma' \rightarrow \mathbb{C}$  from the mirror curve  $\Sigma'$  associated to the Hurwitz numbers, they are able to show the resulting equation, modulo the principle (singular) part, is precisely the equation obtained in the first step.

In this paper, we will give a proof of the BKMP conjecture for the framed vertex  $\mathbb{C}^3$ . In fact, the Bouchard-Marino conjecture is a specialization of the BKMP conjecture for  $\mathbb{C}^3$ , by letting the framing  $f \rightarrow \infty$ . The Hurwitz numbers are then replaced by the Gromov-Witten invariants, which can be written as Hodge integrals involves three  $\lambda$  classes. The generating series of such Hodge integrals satisfies a similar cut-join equation. The corresponding Symmetrized Cut-Join Equation is obtained by the author in a previous paper [3]. Following the line of Eynard-Mulase-Safnuk, we prove the BKMP conjecture by switching the residues calculation to the two nearby simple poles in the question, and pushing forward the Symmetrized Cut-Join Equation of [3] via the projection from the framed mirror curve.

We will describe some known results in Section 2. The BKMP conjecture for  $\mathbb{C}^3$  is stated in Section 3. In Section 4, we compute the residues appeared in the conjectural recursion. Replacing the complex analysis of the functions  $\eta_n(v)$ , invent by Eynard-Mulase-Safnuk, by formal power series argument in

Section 5, we simplify the proof the technical Lemma 5.1, avoid the convergence and analytic continuation difficulty in [4]. Finally, we compute the push forward of the Symmetrized Cut-Join Equation in the last two sections, and establish the BKMP conjecture.

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## 2. MARINO-VAFA FORMULA AND SYMMETRIZED CUT-JOIN EQUATION

The celebrated topological vertex theory developed by Li-Liu-Liu-Zhou [7] establishes a correspondence between two different types of physics theory, topological string theory and the Chern-Simons theory. More precisely, it gives a closed formula of Gromov-Witten invariants of certain toric Calabi-Yau three folds in terms of link invariants (see [8] [9] [7]).

The simplest example is the so-called Marino-Vafa formula proved by Liu-Liu-Zhou [8]. Via virtual localization, one side of the Marino-Vafa formula can be written as a generating series of Hodge integrals involves three lambda classes  $\mathcal{C} = \sum_{g \geq 0, n \geq 1} \mathcal{C}_n^g \lambda^{2g-2+n}$ , where

$$\begin{aligned} \mathcal{C}_n^g &= \sum_{d \geq 1} \sum_{\mu \vdash d, l(\mu)=n} -\frac{\sqrt{-1}^{d+n}}{|\text{Aut } \mu|} (f(1+f))^{n-1} \prod_{i=1}^n \frac{\prod_{a=1}^{\mu_i-1} (\mu_i f + a)}{(\mu_i - 1)!} \int_{\mathcal{M}_{g,n}} \frac{\Gamma_g(f)}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \cdot \mathbf{p}_\mu \\ &= -\frac{\sqrt{-1}^n (f(1+f))^{n-1}}{n!} \sum_{\mu_1, \mu_2, \dots, \mu_n \geq 1}^{+\infty} \sqrt{-1}^{|\mu|} \prod_{i=1}^n \frac{\prod_{a=1}^{\mu_i-1} (\mu_i f + a)}{(\mu_i - 1)!} \\ &\quad \cdot \sum_{b_1, \dots, b_n} \int_{\mathcal{M}_{g,n}} \Gamma_g(f) \prod_{i=1}^n \psi_i^{b_i} \prod_{i=1}^n \mu_i^{b_i} \cdot \mathbf{p}_\mu \\ &= -\frac{(f(1+f))^{n-1}}{n!} \sum_{b_1, \dots, b_n} < \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) > \prod_{i=1}^n \varphi_{b_i}(\vec{\mathbf{p}}). \end{aligned}$$

The formula proved in [8] relates the generating series  $\mathcal{C}$  to a truncated version of the framing dependent Chern-Simons partition function (generating series of colored HOMFLY polynomials) of unknot. In the above formula, we use the notation

$$\varphi_i(\vec{\mathbf{p}}) = \sum_{m \geq 1} \sqrt{-1}^{m+1} \mathbf{p}_m \frac{\prod_{a=1}^{m-1} (mf + a)}{(m-1)!} m^i = \frac{1}{f} \sum_{m \geq 1} \sqrt{-1}^{m+1} \mathbf{p}_m \frac{\prod_{a=0}^{m-1} (mf + a)}{m!} m^i$$

for a formal sum involve infinitely many formal variables  $\vec{\mathbf{p}} = \{\mathbf{p}_1, \mathbf{p}_2, \dots\}$ , and the class

$$\Gamma_g(f) = \Lambda_g^\vee(1) \Lambda_g^\vee(f) \Lambda_g^\vee(-f-1)$$

for

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

The relation between these Hodge integrals and the Gromov-Witten invariants of Calabi-Yau three-folds can be found in [6].

The generating function  $\mathcal{C}$  satisfy a Cut-Join Equation:

$$(2.1) \quad \frac{\partial \mathcal{C}}{\partial f} = \frac{\sqrt{-1} \lambda}{2} \sum_{i,j \geq 1} (ij \mathbf{p}_{i+j} \frac{\partial^2 \mathcal{C}}{\partial \mathbf{p}_i \partial \mathbf{p}_j} + ij \mathbf{p}_{i+j} \frac{\partial \mathcal{C}}{\partial \mathbf{p}_i} \frac{\partial \mathcal{C}}{\partial \mathbf{p}_j} + (i+j) \mathbf{p}_i \mathbf{p}_j \frac{\partial \mathcal{C}}{\partial \mathbf{p}_{i+j}})$$

In [3], the author introduced symmetrization operators

$$\Xi_n : \mathbb{C}[\mathbf{p}_1, \mathbf{p}_2, \dots] \rightarrow \mathbb{C}[x_1, \dots, x_n]$$

with values in the subring of  $\mathbb{C}[x_1, \dots, x_n]$  consists of symmetric polynomials, by letting

$$\Xi_n \mathbf{p}_\alpha = (\sqrt{-1})^{-n-|\alpha|} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\alpha_1} \cdots x_{\sigma(n)}^{\alpha_n}$$

for  $n \geq 1$  if  $l(\alpha) = n$  with  $\alpha = (\alpha_1, \dots, \alpha_n)$ , and 0 otherwise.

After a transcendental change of variables

$$\begin{aligned} y(x)^f(1-y(x)) &= x \\ t &= \frac{1}{(1+f)y-f}, \end{aligned}$$

or explicitly

$$\begin{aligned} t &= 1 + \left(\frac{1+f}{f}\right)((1+f)x \frac{d}{dx}(1-y) - (1-y)) \\ &= 1 + \left(\frac{1+f}{f}\right)\left(\frac{f(1-y)}{(1+f)y-f}\right) \\ &= 1 + \left(\frac{1+f}{f}\right) \sum_{n=1}^{+\infty} \frac{\prod_{a=0}^{n-1}(nf+a)}{n!} x^n, \end{aligned}$$

the generating series become

$$H_{g,n} = -(f(1+f))^{n-1} \sum_{b_1, \dots, b_n} < \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) >_g \prod_{i=1}^n \phi_{b_i}(t_i).$$

In the above expression,

$$\phi_b(t) = \left( \frac{t(t-1)(ft+1)}{f+1} \frac{d}{dt} \right)^b \left( \frac{t-1}{f+1} \right)$$

for  $b \geq 0$  is a polynomial in  $t$  of degree  $2b+1$ , with coefficients in  $\mathbb{Q}(f)$ . For convenience, we denote

$$\phi_{-1}(t) = -\log\left(1 + \frac{1}{ft}\right)$$

for later use. It is easy to check that

$$\phi_0(t) = \frac{t-1}{f+1} = \left( \frac{t(t-1)(ft+1)}{f+1} \frac{d}{dt} \right) \phi_{-1}(t)$$

compatible with our notations.

Apply the symmetrization operator  $\Xi_n$  and change the variables from  $x_i$  into  $t_i$ , Equation 2.1 transformed into the following Symmetrized Cut-Join Equation proved in [3]

$$\left( \frac{\partial}{\partial f} + \sum_{l=1}^n \frac{t_l(t_l-1)}{f+1} \cdot \frac{\partial}{\partial t_l} \right) H_n^g(t_1, \dots, t_n, f) = T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned} T_1 &= -\frac{1}{2} \sum_{l=1}^n t_l(t_l-1) \left( \frac{t_l f+1}{f+1} \right) \frac{\partial}{\partial t_l} \cdot t_{n+1}(t_{n+1}-1) \left( \frac{t_{n+1} f+1}{f+1} \right) \frac{\partial}{\partial t_{n+1}} H_{n+1}^{g-1}|_{t_{n+1}=t_l} \\ T_2 &= -\frac{1}{2} \sum_{1 \leq a \leq g-1} \sum_{1 \leq k \leq n} \Theta_{k-1}(t_1(t_1-1) \left( \frac{t_1 f+1}{f+1} \right) \frac{\partial}{\partial t_1} H_k^a(t_1, \dots, t_k, f)) \\ &\quad \cdot (t_1(t_1-1) \left( \frac{t_1 f+1}{f+1} \right) \frac{\partial}{\partial t_1} H_{n-k+1}^{g-a}(t_1, t_{k+1}, \dots, t_n, f)) \\ T_3 &= -\sum_{k=3}^n \Theta_{k-1}(t_1(t_1-1) \left( \frac{t_1 f+1}{f+1} \right) \frac{\partial}{\partial t_1} H_k^0(t_1, \dots, t_k, f)) (t_1(t_1-1) \left( \frac{t_1 f+1}{f+1} \right) \frac{\partial}{\partial t_1} H_{n-k+1}^g(t_1, t_{k+1}, \dots, t_n, f)) \\ T_4 &= \Theta_1 \frac{t_1^2(t_1-1)(t_2-1)}{t_1-t_2} \left( \frac{t_1 f+1}{f+1} \right)^2 \frac{\partial}{\partial t_1} H_{n-1}^g(t_1, t_3, \dots, t_n, f). \end{aligned}$$

We remark that this is an equality in the polynomial ring  $\mathbb{C}(f)[t_1, \dots, t_n]$ . The symbol  $\Theta_k$  means take the symmetric sum. For more detail about this Symmetrized Cut-Join Equation, we refer to [3].

### 3. THE BKMP CONJECTURE

The Symmetrized Cut-Join Equation is a set of topological recursion of Hodge integrals, i.e, it computes genus  $g$  Hodge integrals in terms of integrals with less genus or less marked points.

The BKMP conjecture is another set of topological recursion of Hodge integrals, coming from the conjectural duality between topological string theory and matrix model theory. We now describe the recursions in the BKMP conjecture for  $\mathbb{C}^3$ . In this case, the Framed Mirror Curve  $C$  is given by

$$x = y^f(1 - y),$$

with the formal power series inverse function

$$y(x) = 1 - \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-2} (nf + j)}{n!} x^n.$$

Denote  $\zeta_n$  (depends on  $f$ ) the differential one form

$$\begin{aligned} \zeta_n &= d(\phi_n) = d\left((x \frac{d}{dx})\left(\frac{t-1}{f+1}\right)\right) \\ &= d\left(\left(\frac{t(t-1)(ft+1)}{f+1}\right)^n \left(\frac{t-1}{f+1}\right)\right) \\ &= d\left(\frac{1}{\tau} \sum_{m=1}^{\infty} \frac{\prod_{a=0}^{m-1} (mf+a)}{m!} m^n x^m\right). \end{aligned}$$

Let  $W_g(x_1, \dots, x_n)$  be the differential  $n$ -form defined by the formula

$$W_g(x_1, \dots, x_n) = (-1)^{g+n} (f(f+1))^{n-1} \sum_{b_1, \dots, b_n} \langle \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) \rangle \prod_{i=1}^n \zeta_{b_i}(y_i(x_i), f).$$

For example

$$\begin{aligned} W_0(x) &= \log y(x) \cdot \frac{dx}{x} \\ W_0(x_1, x_2) &= B(y_1, y_2) - \frac{dx_1 dx_2}{(x_1 - x_2)^2} = \frac{dy_1 dy_2}{(y_1 - y_2)^2} - \frac{dx_1 dx_2}{(x_1 - x_2)^2}. \\ W_0(x_1, x_2, x_3) &= -\frac{f^2}{(f+1)} dt_1 dt_2 dt_3 \end{aligned}$$

If we denote  $d_i = \frac{\partial}{\partial x_i} dx_i = \frac{\partial}{\partial t_i} dt_i$  the differential with respect to the  $i$ -th variable, then the differential  $n$ -form

$$W_g(x_1, \dots, x_n) = (-1)^{g+n-1} d_1 \cdots d_n H_g^n$$

becomes an element in the canonical module  $\mathbb{C}(f)[t_1, \dots, t_n] dt_1 \cdots dt_n$ , if switch to the  $t_i$  variables.

The Framed Mirror Curve  $C$  has a critical point  $(x, y) = (\frac{f^f}{(f+1)^{f+1}}, \frac{f}{f+1})$ . Near the critical point, the morphism

$$C \rightarrow \mathbb{C}$$

sending  $(x, y)$  to  $x$  is (locally) a branching double cover. Let  $q$  and  $\bar{q}$  be two points on the Framed Mirror Curve close to the critical point such that  $x(q) = x(\bar{q})$ . Define one forms

$$\omega(q) = (\log y(q) - \log y(\bar{q})) \frac{dx(q)}{x(q)}$$

$$dE(q, \bar{q}, y_2) = \frac{dy_2}{2} \left( \frac{1}{y_2 - y(q)} - \frac{1}{y_2 - y(\bar{q})} \right).$$

The kernel function is defined to be the formal quotient

$$K(q, \bar{q}, y_2) = \frac{dE(q, \bar{q}, y_2)}{\omega(q)}.$$

The BKMP conjectural recursion reads:

**Conjecture 1** (BKMP). *The differential forms  $W_g(y, y_1, \dots, y_n)$  are completely determined by the following topological recursion relation*

$$\begin{aligned} W_g(y, y_1, \dots, y_n) = & \text{Res}_{y(q)=\frac{f}{f+1}} \frac{dE(q, \bar{q}, y)}{w(q)} [W_{g-1}(y(q), y(\bar{q}), y_1, \dots, y_n) \\ & + \sum_{g_1+g_2=g} \sum_{I \coprod J=H}^{\text{stable}} \sum_{J \subset H} W_{g_1}(y(q), y_I) W_{g_2}(y(\bar{q}), y_J) \\ & + \sum_{i=1}^n (W_g(y(q), y_{H \setminus \{i\}}) \otimes B(y(\bar{q}), y_i) + B(y(q), y_i) \otimes W_g(y(\bar{q}), y_{H \setminus \{i\}}))], \end{aligned}$$

together with the initial conditions

$$W_0(x_1, x_2, x_3) = -\frac{f^2}{(f+1)} dt_1 dt_2 dt_3$$

$$W_1(x_1) = \frac{1}{24} ((1+f+f^2)\zeta_0 - f(1+f)\zeta_1).$$

In the rest of this paper, we will prove that the Symmetrized Cut-Join Equation implies the above BKMP conjecture.

#### 4. RESIDUE CALCULUS

The RHS of the BKMP conjecture involves the following two types of residues.

Type I: Comes from  $W_{g-1}$  and the stable sums.

$$R_{a,b}(y) = \text{Res}_{y(q)=\frac{f}{f+1}} \frac{dE(q, \bar{q}, y)}{\omega(q)} \zeta_a(y(q)) \zeta_b(y(\bar{q})).$$

Type II: Comes from the unstable contribution of integral over the point  $\overline{M}_{0,2}$ .

$$R_a(y, y_i) = \text{Res}_{y(q)=\frac{f}{f+1}} \frac{dE(q, \bar{q}, y)}{\omega(q)} (\zeta_a(y(q)) B(y(\bar{q}), y_i) + \zeta_a(y(\bar{q})) B(y(q), y_i)).$$

Before going into the computations, we first explain the meaning of the above two residues. Take a small open neighborhood  $U$  of  $\frac{f^f}{(f+1)^{f+1}}$  in  $\mathbb{C}$ , such that the projection

$$\pi : \mathbb{C} \rightarrow \mathbb{C}$$

is two to one on the open set  $\pi^{-1}(U \setminus \{\frac{f^f}{(f+1)^{f+1}}\})$ . Denote by  $s : q \mapsto \bar{q}$  the holomorphic inversion, which is well defined on the open neighborhood  $\pi^{-1}U$  of the critical point.

In the formula of Type I case,

$$\begin{aligned}\zeta_a(y(q)) &= \phi'_a(y(q))dx(q) \\ \zeta_b(y(\bar{q})) &= \phi'_b(y(\bar{q}))s'(x(q))dx(q)\end{aligned}$$

where  $\phi'_n = \frac{d}{dx}\phi_n$  is the derivative of  $\phi_n$ . The denominator  $w(q)$  of the formal quotient  $K(q, \bar{q}, y_2)$  canceled one of the  $dx(q)$ , thus the product  $K(q, \bar{q}, y)\zeta_a(y(q))\zeta_b(y(\bar{q}))$  is a two form. After taking the residue,  $R_{a,b}(y)$  became a meromorphic one form. Similar description holds for the two form  $R_a(y, y_i)$ .

**Lemma 4.1.** *Change into the variable  $t$ , i.e, let  $R_{a,b}(y) = P_{a,b}(t)dt$ , then  $P_{a,b}(t) \in \mathbb{C}(f)[t]$  is a polynomial of  $t$  with coefficients in the field  $\mathbb{C}(f)$ .*

*Proof.* Let  $y = \frac{f+\frac{1}{t}}{f+1}$  and let  $z = \frac{1}{t}$ . The critical point of the Framed Mirror Curve is then the point 0 in the  $z$  coordinate. Formally, the kernel function  $K(q, \bar{q}, y)$  is the product of  $\frac{(f+1)dy}{2} \otimes \frac{x(q)}{dx(q)}$  and the function

$$\frac{(\frac{1}{z-z(q)} - \frac{1}{z-z(\bar{q})})}{\log(1 + \frac{z(q)}{f}) - \log(1 + \frac{z(\bar{q})}{f})} = \frac{1}{(z - z(q))(z - z(\bar{q}))} \cdot \left[ \frac{z(q) - z(\bar{q})}{\log(1 + \frac{z(q)}{f}) - \log(1 + \frac{z(\bar{q})}{f})} \right]$$

holomorphic around  $z(q) = 0$ .

We abuse the notation

$$\zeta_a(y(q)) = \zeta_a(t(q)) = \zeta_a\left(\frac{1}{z(q)}\right),$$

which is in the module of differentials  $\mathbb{C}(f)[t(q)]dt(q)$ .

Recall that

$$\frac{x(q)}{dx(q)} = \frac{t(q)(t(q) - 1)(ft(q) + 1)}{(f + 1)dt(q)} = -\frac{(1 - z)(f + z)}{z(f + 1)dz}.$$

By linearity, we only need to consider the residue

$$\text{Res}_{z(q)=0} \frac{F(z(q))}{(z - z(q))(z - z(\bar{q}))} \cdot \frac{1}{z(q)^a} \frac{dz(q)}{z(q)^5} \otimes dy$$

for integer  $a \geq 0$  and functions  $F(z)$  holomorphic around  $z = 0$ . Since  $dy = -\frac{dt}{t^2(f+1)}$ , we have

$$-\frac{dt}{f+1} \cdot \text{Res}_{z(q)=0} \left( F(z(q)) \cdot \sum_{k=0}^{+\infty} \frac{1}{z^k} \left( \sum_{i=0}^k z(q)^i z(\bar{q})^{k-i} \right) \frac{dz(q)}{z(q)^{5+a}} \right)$$

by expanding the denominator. This expression only contains non-negative power of  $\frac{1}{z} = t$ . Moreover, if  $k \geq 5 + a$ , then the summand is then holomorphic at  $z(q) = 0$ , and has no residue. This shows that the above residue is a polynomial times  $dt$ . A more careful analysis of the parameter  $f$  in the above process shows that the coefficients are in fact rational functions in the variable  $f$ .  $\square$

Under the  $z$  coordinate, the projection  $\pi : \mathbb{C} \rightarrow \mathbb{C}$  maps the critical point to  $z = 0$ . By shrinking the open neighborhood  $U$  suitably, we can find a simple closed curve around  $z = 0$ . The holomorphic inversion  $s$  is thus defined in the interior of the area bounded by  $\gamma$ . Let the variable  $y$  sufficiently close to the critical point such that it is inside the curve  $\gamma$ .

Let  $\zeta_a(y) = F_a(t)dt$  for some polynomial  $F_a(t) \in \mathbb{C}(f)[t]$ . By compactness, the functions  $F_a(y(q))$ ,  $F_b(y(\bar{q}))$ ,  $\frac{1}{\log(y(q)) - \log(y(\bar{q}))}$  and  $t(q)(t(q) - 1)(ft(q) + 1)$  are all bounded along the circle  $\gamma$ . We thus have

the following estimate of the contour integral:

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{\gamma} \frac{dE(q, \bar{q}, y)}{\omega(q)} \zeta_a(y(q)) \zeta_b(y(\bar{q})) \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{\frac{1}{2}(\frac{1}{y-y(q)} - \frac{1}{y-y(\bar{q})})}{\log y(q) - \log y(\bar{q})} \cdot \left( \frac{x(q)}{dx(q)} \otimes dy \right) \cdot F_a(t(q)) dt(q) \cdot F_b(t(\bar{q})) dt(\bar{q}) \\
&= \frac{dy}{2\pi i} \int_{\gamma} \frac{\frac{1}{2}(\frac{1}{y-y(q)} - \frac{1}{y-y(\bar{q})})}{\log y(q) - \log y(\bar{q})} \cdot \frac{t(q)(t(q)-1)(ft(q)+1)}{(f+1)} \cdot F_a(t(q)) \cdot F_b(t(\bar{q})) s'(t(q)) dt(q) \\
&\leq M \cdot \frac{dt}{|t^2|} \sim \frac{dt}{|t^2|}.
\end{aligned}$$

By hypothesis, we choose  $z$  small enough (sufficiently close to the critical point), such that it is in the interior bounded by  $\gamma$ . Then we choose another simple closed curve  $\gamma_{\infty}$  around  $z = 0$  (i.e,  $t = \infty$ ), such that both  $z$  and  $s(z)$  are outside  $\gamma_{\infty}$ .

The above estimate together with Lemma 4.1 imply that the above integral is the principle part of the following integral

$$A := \frac{1}{2\pi i} \int_{\gamma \cup -\gamma_{\infty}} \frac{dE(q, \bar{q}, y)}{\omega(q)} \zeta_a(y(q)) \zeta_b(y(\bar{q})).$$

There are two poles  $z(q) = z$  and  $z(q) = s(z)$  between the two circle  $\gamma$  and  $\gamma_{\infty}$ . The integral  $A$  thus can be computed by residue theorem:

$$\begin{aligned}
A &= \text{Res}_{z(q)=z, s(z)} \frac{\frac{f+1}{2}(\frac{1}{z-z(q)} - \frac{1}{z-z(\bar{q})})}{\log(1 + \frac{z(q)}{f}) - \log(1 + \frac{z(\bar{q})}{f})} dy \zeta_a(\frac{1}{z(q)}) \zeta_b(\frac{1}{z(\bar{q})}) \cdot \frac{x(q)}{dx(q)} \\
&= \text{Res}_{z(q)=z, s(z)} \frac{\frac{z^2 dt}{2}(\frac{1}{z-z(q)} - \frac{1}{z-z(\bar{q})})}{\log(1 + \frac{z(q)}{f}) - \log(1 + \frac{z(\bar{q})}{f})} \cdot F_a(\frac{1}{z(q)}) F_b(\frac{1}{z(\bar{q})}) s'(\frac{1}{z(q)}) \frac{dz(q)}{z(q)^2} \cdot \frac{(1-z(q))(z(q)+f)}{(f+1)z(q)^3} \\
&= \frac{-\frac{1}{2} dt}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} F_a(t) F_b(s(t)) s'(t) \cdot \frac{t(t-1)(ft+1)}{(f+1)} \\
&+ \frac{-\frac{1}{2} dt}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} F_a(s(t)) F_b(t) \cdot \frac{s(t)(s(t)-1)(fs(t)+1)}{(f+1)} \\
&= \frac{-\frac{1}{2}}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} (\phi_{a+1}(t) d\phi_b(s(t)) + \phi_{a+1}(s(t)) d\phi_b(t)) \\
&= \frac{-\frac{1}{2}}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} (\phi_{a+1}(t) \phi_{b+1}(s(t)) + \phi_{a+1}(s(t)) \phi_{b+1}(t)) \cdot \frac{(f+1)dt}{t(t-1)(ft+1)}.
\end{aligned}$$

Let  $f(t) = \sum_{n \geq -N}^{\infty} a_n t^n \in \mathbb{C}(f)((t))$  be a Laurent series. Denote by  $f(t)_+ = \sum_{n \geq 0}^{\infty} a_n t^n$  the regular part of  $f(t)$ . The above formula of  $A$  and the estimate of the contour integral along  $\gamma$  give the following proposition.

**Proposition 4.2.**

$$P_{a,b}(t) = \left( \frac{\frac{1}{2}}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} (\phi_{a+1}(t) \phi_{b+1}(s(t)) + \phi_{a+1}(s(t)) \phi_{b+1}(t)) \cdot \frac{(f+1)}{t(t-1)(ft+1)} \right)_+$$

Next we consider the Type II residues. A similar argument as in the Type I case leads to the following lemma, the proof of which we left for the reader.

**Lemma 4.3.** *Change to the  $t$  and  $t_i$  variable,  $R_a(y, y_i) = P_a(t, t_i) dt dt_i$ , then  $P_a(t, t_i) \in \mathbb{C}(f)[t, t_i]$  is a polynomial of two variables  $t$  and  $t_i$  with coefficients in the field  $\mathbb{C}(f)$ .*

Let the variable  $t_i$  take values in the area such that its inverse  $z_i$  and  $s(z_i)$  are outside the area bounded by  $\gamma$ . Let us now consider the following integral

$$B := \frac{1}{2\pi i} \int_{\gamma \sqcup -\gamma_\infty} \frac{dE(q, \bar{q}, y)}{w(q)} (\zeta_z(y(q))B(y(\bar{q}), y_i) + \zeta_a(y(\bar{q}))B(y(q), y_i)).$$

We expect the integral along  $\gamma$  is the principle part, while the  $\gamma_\infty$  part gives the residue we compute.

By our hypothesis, there are two simple poles in the area bounded by the contour, so by residue theorem:

$$\begin{aligned} B &= \frac{1}{2\pi i} \int_{\gamma - \gamma_\infty} \frac{dE(q, \bar{q}, y)}{w(q)} (\zeta_n(y(q))B(y(\bar{q}), y_i) + \zeta_n(y(\bar{q}))B(y(q), y_i)) \\ &= \text{Res}_{z(q)=t^{-1}, s(t)^{-1}} \frac{\frac{f+1}{2}(\frac{1}{z-z(q)} - \frac{1}{z-z(\bar{q})})}{\log(1 + \frac{z(q)}{f}) - \log(1 + \frac{z(\bar{q})}{f})} dy dy_i \cdot \frac{t(q)(t(q)-1)(ft(q)+1)}{(f+1)} \\ &\quad \cdot \left( F_n(t(q)) \frac{dy(\bar{q})}{(y(\bar{q})-y_i)^2} + F_n(t(\bar{q})) \frac{s'(t(q))dy(q)}{(y(q)-y_i)^2} \right) \\ &= \frac{-s'(t)dt dt_i}{2(\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)}))} \cdot \frac{t(t-1)(ft+1)}{(f+1)} \cdot \left( \frac{F_n(t)}{(s(t)-t_i)^2} + \frac{F_n(s(t))}{(t-t_i)^2} \right) \\ &\quad + \frac{-dt dt_i}{2(\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)}))} \cdot \frac{s(t)(s(t)-1)(fs(t)+1)}{(f+1)} \cdot \left( \frac{F_n(s(t))}{(t-t_i)^2} + \frac{F_n(t)}{(s(t)-t_i)^2} \right) \\ &= \frac{-s'(t)dt dt_i}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} \cdot \frac{t(t-1)(ft+1)}{(f+1)} \cdot \left( \frac{F_n(t)}{(s(t)-t_i)^2} + \frac{F_n(s(t))}{(t-t_i)^2} \right) \\ &= \frac{-dt dt_i}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} \cdot \left( \frac{\phi_{n+1}(t)s'(t)}{(s(t)-t_i)^2} + \frac{\phi_{n+1}(s(t))}{(t-t_i)^2} \right) \\ &= - \frac{\phi_{n+1}(t)B(s(t), t_i) + \phi_{n+1}(s(t))B(t, t_i)}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})}. \end{aligned}$$

After a more careful examination of the expression  $B$ , by expand it as a formal Laurent series, it is not hard to see that in fact  $B \in \mathbb{C}(f)[t_i]((t))$ . For a Laurent series

$$f(t, t_i) = \sum_{n \geq -N}^{\infty} a_n(t_i) t^n \in \mathbb{C}(f)[t_i]((t)),$$

we denote by  $f(t, t_i)_+$  the truncation  $\sum_{n \geq 0}^{\infty} a_n(t_i) t^n \in \mathbb{C}(f)[t_i][[t]]$  of the regular part of  $f(t, t_i)$ . By a similar argument as we have done for the Type I residue, we have the estimate

$$\left| \frac{1}{2\pi i} \int_{\gamma} \frac{dE(q, \bar{q}, y)}{w(q)} (\zeta_a(y(q))B(y(\bar{q}), y_i) + \zeta_a(y(\bar{q}))B(y(q), y_i)) \right| \leq M \cdot \frac{dt dt_i}{|t^2|} \sim \frac{dt dt_i}{|t^2|},$$

which has to be the principle part of the Laurent series, according to Lemma 4.3. This finishes the proof of the following

**Proposition 4.4.**

$$P_n(t, t_i) = \left( \frac{\phi_{n+1}(t)B(s(t), t_i) + \phi_{n+1}(s(t))B(t, t_i)}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} \right)_+$$

The calculation of this section proves the following theorem:



**Theorem 4.5.** *The topological recursion in the BKMP conjecture 1 is equivalent to*

$$\begin{aligned}
& (f(f+1))^{n-1} \sum_{b_1, \dots, b_n} < \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) >_g \prod_{k=1}^n d\phi_{b_k}(t_k) \\
& = (f(f+1))^n \sum_{a_1, a_2, b_2, \dots, b_n} < \tau_{a_1} \tau_{a_2} \prod_{k=2}^n \Gamma_{g-1}(f) >_{g-1} \prod_{k=2}^n d\phi_{b_k}(t_k) \cdot P_{a_1, a_2}(t_1) dt_1 \\
& \quad - (f(f+1))^{n-1} \sum_{\substack{\text{stable} \\ g_1+g+2=g, I \coprod J = \{2, \dots, n\}}} \sum_{a_1, a_2, b_2, \dots, b_n} < \tau_{a_1} \prod_{i \in I} \tau_{b_i} \Gamma_{g_1}(f) >_{g_1} \\
& \quad \cdot < \tau_{a_2} \prod_{j \in J} \tau_{b_j} \Gamma_{g_2}(f) >_{g_2} \prod_{k=2}^n d\phi_{b_k}(t_k) \cdot P_{a_1, a_2}(t_1) dt_1 \\
& \quad - (f(f+1))^{n-2} \sum_{j=2}^n \sum_{b, b_i, i \in \{2, \dots, n\} \setminus \{j\}} < \tau_b \prod_{k=2, k \neq j}^n \tau_{b_k} > \prod_{k=2, k \neq j}^n d\phi_{b_k}(t_k) \cdot P_b(t_1, t_j).
\end{aligned}$$

## 5. SOME FORMAL ANALYSIS

The critical point of the curve  $x = y^f(1-f)$  is

$$(x, y) = \left( \frac{f^f}{(f+1)^{f+1}}, \frac{f}{f+1} \right).$$

Let

$$x = \frac{f^f}{(f+1)^{f+1}} e^{-\frac{f+1}{f}w} \quad \text{and} \quad w = \frac{1}{2}v^2,$$

then we have

$$v^2 = z^2 \left( 1 + \sum_{k=1}^{+\infty} \frac{z^k}{k+2} \left( \frac{1 - (\frac{-1}{f})^{k+1}}{1 - (\frac{-1}{f})} \right) \right) \in \mathbb{Q}(f)[[z]].$$

Let  $F(z)$  be the unique formal power series in  $\mathbb{Q}(f)[[z]]$  such that  $F(0) = 1$  and

$$F(z)^2 = \left( 1 + \sum_{k=1}^{+\infty} \frac{z^k}{k+2} \left( \frac{1 - (\frac{-1}{f})^{k+1}}{1 - (\frac{-1}{f})} \right) \right).$$

The coefficients of  $F(z)F(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$  are determined recursively by the relations

$$\begin{aligned}
a_0 &= 1 \\
\sum_{i=0}^k a_i a_{k-i} &= \frac{1}{k+2} \left( \frac{1 - (\frac{-1}{f})^{k+1}}{1 + \frac{1}{f}} \right) \quad \text{for } k \geq 1.
\end{aligned}$$

It is obvious that the two possible solutions of  $v$  are  $\pm zF(z)$ . We take  $v$  to be the solution  $zF(z)$ , which admit a formal inverse function

$$z = vG(v) = v + \sum_{k=1}^{\infty} g_k v^{k+1} \in v\mathbb{Q}(f)[[v]].$$

The holomorphic map  $s$  can be described as by sending  $t = \frac{1}{vG(v)}$  to  $s(t) = \frac{1}{-vG(-v)}$ , i.e, sending  $v \mapsto -v$ . It is easy to see that

$$t = \frac{1}{v[1 + \sum_{k=1}^{\infty} g_k v^k]} = \frac{1}{v} [1 + \sum_{k=1}^{\infty} h_k v^k] \in v^{-1}\mathbb{Q}(f)[[v]].$$

**Lemma 5.1.** *Let*

$$\eta_{-1}(v) = -\frac{1}{2}(\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})) = \frac{1}{2}(\phi_{-1}(t) - \phi_{-1}(s(t))),$$

and let  $\eta_{n+1}(v) = -\frac{f}{f+1} \frac{1}{v} \frac{d}{dv} \eta_n(v)$  for all  $n \geq -1$ , then we have

$$\eta_n(v) = \frac{1}{2}(\phi_n(t) - \phi_n(s(t))).$$

Moreover, for each  $n \geq -1$

$$\eta_n(v) = \phi_n(t) + F_n(w)$$

for some  $F_n(w) \in \mathbb{Q}(f)[[w]]$ .

*Proof.* By definition, we have

$$-\frac{f}{f+1} \cdot \frac{1}{v} \frac{d}{dv} = -\frac{f}{f+1} \cdot \frac{d}{dw} = x \frac{d}{dx} = \frac{t(t-1)(ft+1)}{(f+1)} \frac{d}{dt}$$

is invariant under the action of  $s : t \mapsto s(t)$ . Applying this operator  $n$  times to the definition of  $\eta_{-1}(v)$ , we get the first claim.

It is obvious that  $\eta_n(v)$  is odd in the variable  $v$ , so  $\eta_{-1}(v) = \frac{-v}{f}(1 + \dots) \in v\mathbb{Q}(f)[[v]]$  contains only odd power of  $v$ . By applying the operator  $-\frac{f}{f+1} \frac{1}{v} \frac{d}{dv}$  to  $\eta_{-1}(v)$ , we get

$$\eta_n(v) = \frac{f^n}{(f+1)^{n+1}} \cdot \frac{(2n-1)!!}{v^{2n+1}} (1 + \dots) \in \frac{1}{v^{2n+1}} \mathbb{Q}(f)[[w]].$$

We have

$$\begin{aligned} \eta_{-1}(v) - \phi_{-1}(t) &= -\frac{1}{2}(\phi_{-1}(t) + \phi_{-1}(s(t))) \\ &= \frac{1}{2} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{nf^n} (v^n G(v)^n + (-v)^n G(-v)^n), \end{aligned}$$

which is in the formal power series ring  $\mathbb{Q}(f)[[v]]$ . Moreover, it is even in the variable  $v$ , thus in the ring  $\mathbb{Q}(f)[[w]]$ . We denote it by  $F_{-1}(w)$ . Since there is no constant term,  $F_{-1}(w) \in \mathbb{Q}(f)[[w]]$ . This proves the second claim for  $n = -1$ .

For  $n > -1$ , apply the operator  $-\frac{f}{f+1} \frac{1}{v} \frac{d}{dv}$  to the equation

$$\eta_{-1}(v) - \phi_{-1}(t) = F_{-1}(w)$$

repeatedly  $n$  times, we obtain

$$\eta_n(v) = \phi_n(t) + F_n(w)$$

for  $F_n(w) = (-\frac{f+1}{f} \frac{d}{dw})^n F_{-1}(w) \in \mathbb{Q}(f)[[w]]$  by induction. □

**Corollary 5.2.**

$$P_{a,b} dt = -\frac{1}{2} \left( \frac{\eta_{a+1}(v)\eta_{b+1}(v)}{\eta_{-1}(v)} \left( \frac{f+1}{f} \right) v dv|_{v=v(t)} \right)_+$$

*Proof.* By Proposition 4.2, we have

$$\begin{aligned} & \frac{\frac{1}{2}}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} (\phi_{a+1}(t)\phi_{b+1}(s(t)) + \phi_{a+1}(s(t))\phi_{b+1}(t)) \cdot \frac{(f+1)dt}{t(t-1)(ft+1)} \\ &= \frac{1}{2\eta_{-1}(v)} \cdot \frac{(f+1)dt}{t(t-1)(ft+1)} \cdot \left( \frac{\phi_{a+1}(t) - \phi_{a+1}(s(t))}{2} \frac{\phi_{b+1}(t) - \phi_{b+1}(s(t))}{2} \right. \\ & \quad \left. - \frac{\phi_{a+1}(t) + \phi_{a+1}(s(t))}{2} \frac{\phi_{b+1}(t) + \phi_{b+1}(s(t))}{2} \right) \\ &= -\frac{(f+1)v dv}{2f\eta_{-1}(v)} (\eta_{a+1}(v)\eta_{b+1}(v) - F_{a+1}(w)F_{b+1}(w)). \end{aligned}$$

To complete the proof, only need to notice that

$$\left( \frac{F_{a+1}(w)F_{b+1}(w)vdv}{\eta_{-1}(v)} \Big|_{v \mapsto v(t)} \right)_+ = 0.$$

□

## 6. THE LEFT HAND SIDE

After all the preparations, we are now ready to prove the BKMP conjecture. We will show that the push forward of the Symmetrized Cut-Join Equation via the first variable from the Framed Mirror Curve to  $\mathbb{C}$  gives the BKMP conjecture.

In this section, we will deal with the LHS of the Symmetrized Cut-Join Equation. Unlike the Hurwitz number case studied in [4], the LHS of our Symmetrized Cut-Join Equation involves taking derivative with respect to the parameter  $f$  that need more special treatment.

The LHS of the symmetrized cut-join equation reads

$$\begin{aligned} & \left( \frac{\partial}{\partial f} \Big|_{t_i} + \sum_{l=1}^n \frac{t_l(t_l - 1)}{f + 1} \cdot \frac{\partial}{\partial t_l} \right) H_n^g(t_1, \dots, t_n, f) \\ &= \frac{\partial}{\partial f} \Big|_{y_i} H_n^g(t_1(y_1, f), \dots, t_n(y_n, f), f) \\ &= \left( \frac{\partial}{\partial f} \Big|_{x_i} + \sum_{l=1}^n \log y_l \cdot x_l \frac{\partial}{\partial x_l} \right) H_n^g. \end{aligned}$$

If we change the variable

$$x = \frac{f^f}{(f+1)^{f+1}} e^{-(\frac{f+1}{2f})v^2},$$

then

$$\frac{dx}{df} = x \left( \frac{f}{(f+1)^2} + \frac{v^2}{2f^2} + \log\left(\frac{f}{f+1}\right) \right),$$

and the differential operator above then becomes

$$\begin{aligned} & \frac{\partial}{\partial f} \Big|_{x_i} + \sum_{l=1}^m \log y_l \cdot x_l \frac{\partial}{\partial x_l} \\ &= \frac{\partial}{\partial f} \Big|_{v_i} - \sum_{l=1}^m \left( \log\left(1 + \frac{1}{ft_l}\right) - \frac{f}{(f+1)^2} - \frac{v_l^2}{2f^2} \right) \frac{f}{v_l(f+1)} \frac{\partial}{\partial v_l}. \end{aligned}$$

Taking the direct image of a function  $F$  on the framed curve via the map  $\pi : C \rightarrow \mathbb{C}$  with respect to the first variable, i.e,  $\pi_*(F) = F(v_1) + F(-v_1)$ . The term with  $\frac{\partial}{\partial f}$  becomes

$$\begin{aligned} & \frac{\partial}{\partial f} \Big|_{v_i} \left( -(f(f+1))^{n-1} \sum_{b_1, \dots, b_n} \langle \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) \rangle_g (\phi_{b_1}(t_1) + \phi_{b_1}(s(t_1))) \prod_{k=2}^n \phi_{b_k}(t_k) \right) \\ &= \frac{\partial}{\partial f} \Big|_{v_i} \left( -2(f(f+1))^{n-1} \sum_{b_1, \dots, b_n} \langle \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) \rangle_g F_{b_1}(w_1) \prod_{k=2}^n \phi_{b_k}(t_k) \right), \end{aligned}$$

which we denoted by  $\tilde{A}$  for short. In the summation, the  $l \neq 1$  term becomes a sum of

$$\left( \log\left(1 + \frac{1}{ft_l}\right) - \frac{f}{(f+1)^2} - \frac{v_l^2}{2f^2} \right) \frac{f}{(f+1)v_l} \frac{\partial}{\partial v_l} \left( F_{w_1}(t_1) \prod_{k=2}^n \phi_{b_k}(t_k) \right),$$

which we denote by  $A_l(b_1, \dots, b_n)$ . The term with  $l = 1$  contributes a sum of

$$\begin{aligned}
& \left( \log\left(1 + \frac{1}{ft_1}\right) - \frac{f}{(f+1)^2} - \frac{v_1^2}{2f^2} \right) \frac{f}{(f+1)v_1} \frac{\partial}{\partial v_1} (\eta_{b_1}(v_1) - F_{b_1}(w_1)) \prod_{k=2}^n \phi_{b_k}(t_k) \\
& + \left( \log\left(1 + \frac{1}{fs(t_1)}\right) - \frac{f}{(f+1)^2} - \frac{v_1^2}{2f^2} \right) \frac{f}{(f+1)v_1} \frac{\partial}{\partial v_1} (\eta_{b_1}(-v_1) - F_{b_1}(w_1)) \prod_{k=2}^n \phi_{b_k}(t_k) \\
& = \left( \log\left(1 + \frac{1}{ft_1}\right) - \log\left(1 + \frac{1}{fs(t_1)}\right) \right) \frac{f}{(f+1)v_1} \frac{\partial}{\partial v_1} \eta_{b_1}(v_1) \prod_{k=2}^n \phi_{b_k}(t_k) \\
& + \left( \log\left(1 + \frac{1}{ft_1}\right) + \log\left(1 + \frac{1}{fs(t_1)}\right) - \frac{2f}{(f+1)^2} - \frac{v_1^2}{f^2} \right) F_{b_1+1}(w_1) \prod_{k=2}^n \phi_{b_k}(t_k) \\
& = 2\eta_{-1}(v_1)\eta_{b_1+1}(v_1) \prod_{k=2}^n \phi_{b_k}(t_k) + A_1(b_1, \dots, b_n),
\end{aligned}$$

where  $A_1(b_1, \dots, b_n)$  denote the part involves  $F_{b_1+1}(w_1)$ .

In summary, the push forward of the LHS of the Symmetrized Cut-Join Equation is the following:

$$\begin{aligned}
L &:= \tilde{A} + (f(f+1))^{n-1} \sum_{b_1, \dots, b_n} \langle \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) \rangle_g \\
&\quad \cdot \left( 2\eta_{-1}(v_1)\eta_{b_1+1}(v_1) \prod_{k=2}^n \phi_{b_k}(t_k) + \sum_{l=1}^n A_l(b_1, \dots, b_n) \right) \\
&= 2(f(f+1))^{n-1} \sum_{b_1, \dots, b_n} \langle \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) \rangle_g \eta_{-1}(v_1)\eta_{b_1+1}(v_1) \prod_{k=2}^n \phi_{b_k}(t_k) + \tilde{B},
\end{aligned}$$

where we write

$$\tilde{B} := \tilde{A} + (f(f+1))^{n-1} \sum_{b_1, \dots, b_n} \langle \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) \rangle_g \sum_{l=1}^n A_l(b_1, \dots, b_n).$$

for short. The crucial observation is that  $\tilde{A}$  and  $A_l(b_1, \dots, b_n)$  for  $l = 1, \dots, n$  are all regular in  $w_1$ . This can be check from their definition. Thus their sum  $\tilde{B}$  is also regular in  $w_1$ , and we have the important estimate

$$\left( \frac{\tilde{B}}{2\eta_{-1}(v_1)} \cdot \frac{(f+1)dt}{t(t-1)(ft+1)} \right)_+ = 0.$$

On the other hand, we have

$$\begin{aligned}
W_g &= (-1)^{g+n} (f(f+1))^{n-1} \sum_{b_1, \dots, b_n} \langle \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) \rangle_g \prod_{k=1}^n d\phi_{b_k}(t_k) \\
&= (-1)^{g+n} \frac{(f+1)dt_1}{2\eta_{-1}(v_1)t_1(t_1-1)(ft_1+1)} d_2 \cdots d_n \\
&\quad \cdot \left( L - \tilde{B} + (f(f+1))^{n-1} \sum_{b_1, \dots, b_n} \langle \tau_{b_1} \cdots \tau_{b_n} \Gamma_g(f) \rangle_g F_{b_1+1}(w_1) \prod_{k=2}^n \phi_{b_k}(t_k) \right) \\
&= (-1)^{g+n} \left( \frac{(f+1)dt_1}{2\eta_{-1}(v_1)t_1(t_1-1)(ft_1+1)} d_2 \cdots d_n L \right)_+
\end{aligned}$$

This finishes the computation of the LHS of the BKMP conjecture.

## 7. THE RIGHT HAND SIDE

We now compute the push forward of the RHS of the Symmetrized Cut-Join Equation. Recall that it is the sum of the following four terms:

$$\begin{aligned}
T_1 &= -\frac{1}{2} \sum_{l=1}^n t_l(t_l-1) \left( \frac{t_l f + 1}{f+1} \right) \frac{\partial}{\partial t_l} \cdot t_{n+1}(t_{n+1}-1) \left( \frac{t_{n+1} f + 1}{f+1} \right) \frac{\partial}{\partial t_{n+1}} H_{n+1}^{g-1} |_{t_{n+1}=t_l} \\
T_2 &= -\frac{1}{2} \sum_{1 \leq a \leq g-1} \sum_{1 \leq k \leq n} \Theta_{k-1}(t_1(t_1-1) \left( \frac{t_1 f + 1}{f+1} \right) \frac{\partial}{\partial t_1} H_k^a(t_1, \dots, t_k, f)) \\
&\quad \cdot (t_1(t_1-1) \left( \frac{t_1 f + 1}{f+1} \right) \frac{\partial}{\partial t_1} H_{n-k+1}^{g-a}(t_1, t_{k+1}, \dots, t_n, f)) \\
T_3 &= -\sum_{k=3}^n \Theta_{k-1}(t_1(t_1-1) \left( \frac{t_1 f + 1}{f+1} \right) \frac{\partial}{\partial t_1} H_k^0(t_1, \dots, t_k, f)) (t_1(t_1-1) \left( \frac{t_1 f + 1}{f+1} \right) \frac{\partial}{\partial t_1} H_{n-k+1}^g(t_1, t_{k+1}, \dots, t_n, f)) \\
T_4 &= \Theta_1 \frac{t_1^2(t_1-1)(t_2-1)}{t_1-t_2} \left( \frac{t_1 f + 1}{f+1} \right)^2 \frac{\partial}{\partial t_1} H_{n-1}^g(t_1, t_3, \dots, t_n, f).
\end{aligned}$$

The push forward of the  $T_1$  part of the RHS of the Symmetrized Cut-Join Equation is the sum of

$$\begin{aligned}
&\frac{1}{2} (f(f+1))^n \sum_{b_1, \dots, b_{n+1}} \langle \tau_{b_1} \cdots \tau_{b_{n+1}} \Gamma_{g-1}(f) \rangle_{g-1} \\
&\quad \cdot (\phi_{b_1+1}(t_1) \phi_{b_{n+1}+1}(t_1) + \phi_{b_1+1}(s(t_1)) \phi_{b_{n+1}+1}(s(t_1))) \cdot \prod_{k=2}^n \phi_{b_k}(t_k) \\
&= (f(f+1))^n \sum_{b_1, \dots, b_{n+1}} \langle \tau_{b_1} \cdots \tau_{b_{n+1}} \Gamma_{g-1}(f) \rangle_{g-1} \\
&\quad \cdot (\eta_{b_1+1}(v_1) \eta_{b_{n+1}+1}(v_1) + F_{b_1+1}(w_1) F_{b_{n+1}+1}(w_1)) \cdot \prod_{k=2}^n \phi_{b_k}(t_k)
\end{aligned}$$

and

$$-\sum_{l=2}^m (f(f+1))^n \sum_{b_1, \dots, b_{n+1}} \langle \tau_{b_1} \cdots \tau_{b_{n+1}} \Gamma_{g-1}(f) \rangle_{g-1} F_{b_1}(w_1) (\phi_{b_l+1}(t_l) \phi_{b_{n+1}+1}(t_l)) \prod_{i=2, i \neq l}^n \phi_{b_i}(t_i).$$

Thus we have

$$\begin{aligned}
&\left( \frac{\pi_* T_1 \cdot (f+1) dt_1}{2\eta_{-1}(v_1) t_1(t_1-1)(f t_1 + 1)} \right)_+ \\
&= (f(f+1))^n \sum_{b_1, \dots, b_{n+1}} \langle \tau_{b_1} \cdots \tau_{b_{n+1}} \Gamma_{g-1}(f) \rangle_{g-1} \prod_{k=2}^n \phi_{b_k}(t_k) \\
&\quad \cdot \left( -\frac{(f+1) \eta_{b_1+1}(v_1) \eta_{b_{n+1}+1}(v_1) v_1 dv_1}{2f \eta_{-1}(v_1)} \right)_+ \\
&= (f(f+1))^n \sum_{b_1, \dots, b_{n+1}} \langle \tau_{b_1} \cdots \tau_{b_{n+1}} \Gamma_{g-1}(f) \rangle_{g-1} \prod_{k=2}^n \phi_{b_k}(t_k) \cdot P_{b_1, b_{n+1}}(t_1) dt_1.
\end{aligned}$$

By completely the same computation, applied to  $T_2 + T_3$ , we have

$$\begin{aligned} & \left( \frac{\pi_*(T_2 + T_3) \cdot (f+1)dt_1}{2\eta_{-1}(v_1)t_1(t_1-1)(ft_1+1)} \right)_+ \\ &= - (f(f+1))^{n-1} \sum_{g_1+g_2=g} \sum_{I \coprod J=\{2,\dots,n\}}^{stable} \sum_{a_1,a_2,b_2,\dots,b_n} < \tau_{a_1} \prod_{i \in I} \tau_{b_i} \Gamma_{g_1}(f) >_{g_1} \\ & \quad \cdot < \tau_{a_2} \prod_{j \in J} \tau_{b_j} \Gamma_{g_2}(f) >_{g_2} P_{a_1,a_2}(t_1) dt_1 \prod_{k=2}^n \phi_{b_k}(t_k). \end{aligned}$$

The  $T_4$  part is much more complicated, and need additional care. First of all, it is equal to

$$- \sum_{1 \leq i,j \leq n} \frac{t_i(ft_i+1)(t_j-1)}{(f+1)(t_i-t_j)} (f(f+1))^{n-2} \sum_{b,b_i,i \in \{1,\dots,n\} \setminus \{i,j\}} < \tau_b \prod_{k=1,k \neq i,j}^n \tau_{b_k} \Gamma_g(f) >_g \phi_{b+1}(t_i) \prod_{k=1,k \neq i,j}^n \phi_{b_k}(t_k).$$

The push forward  $\pi_* T_4 = S_1 + S_2 + S_3$ , for

$$\begin{aligned} S_1 &= \sum_{2 \leq i \neq j \leq n} 2(f(f+1))^{n-2} \frac{t_i(ft_i+1)(t_j-1)}{(f+1)(t_i-t_j)} \sum_{b,b_i,i \in \{1,\dots,n\} \setminus \{i,j\}} < \tau_b \prod_{k=1,k \neq i,j}^n \tau_{b_k} \Gamma_g(f) >_g \\ & \quad \cdot F_{b_1}(w_1) \phi_{b+1}(t_i) \prod_{k=2,k \neq i,j}^n \phi_{b_k}(t_k) \\ S_2 &= - \sum_{j=2}^n (f(f+1))^{n-2} \sum_{b,b_i,i \in \{2,\dots,n\} \setminus \{j\}} < \tau_b \prod_{k=2,k \neq j}^n \tau_{b_k} \Gamma_g(f) >_g \\ & \quad \cdot \left( \frac{t_1(ft_1+1)(t_j-1)}{(f+1)(t_1-t_j)} \phi_{b+1}(t_1) + \frac{s(t_1)(fs(t_1)+1)(t_j-1)}{(f+1)(s(t_1)-t_j)} \phi_{b+1}(s(t_1)) \right) \prod_{k=2,k \neq j}^n \phi_{b_k}(t_k) \\ S_3 &= - \sum_{i=2}^n (f(f+1))^{n-2} \sum_{b,b_i,i \in \{2,\dots,n\} \setminus \{i\}} < \tau_b \prod_{k=2,k \neq i}^n \tau_{b_k} \Gamma_g(f) >_g \\ & \quad \cdot \left( \frac{t_i(ft_i+1)(t_1-1)}{(f+1)(t_i-t_1)} + \frac{t_i(ft_i+1)(s(t_1)-1)}{(f+1)(t_i-s(t_1))} \right) \phi_{b+1}(t_i) \prod_{k=2,k \neq i}^n \phi_{b_k}(t_k). \end{aligned}$$

It is easy to see that

$$\left( \frac{S_1 \cdot (f+1)dt_1}{2\eta_{-1}(v_1)t_1(t_1-1)(ft_1+1)} \right)_+ = 0,$$

since  $S_1$  is holomorphic in the variable  $w_1$ . Because of

$$\begin{aligned} & \frac{t_i(ft_i+1)(t_1-1)}{(f+1)(t_i-t_1)} + \frac{t_i(ft_i+1)(s(t_1)-1)}{(f+1)(t_i-s(t_1))} \\ &= -2 \frac{t_i(ft_i+1)}{f+1} - \frac{t_i(t_i-1)(ft_i)}{f+1} \cdot \sum_{k=0}^{\infty} \left( \frac{t_i^k}{t_1^{k+1}} + \frac{t_i^k}{s(t_1)^{k+1}} \right), \end{aligned}$$

we also have

$$\left( \frac{S_3 \cdot (f+1)dt_1}{2\eta_{-1}(v_1)t_1(t_1-1)(ft_1+1)} \right)_+ = 0.$$

Finally, let us compute the contribution of  $S_2$ . Consider the differentials:

$$\begin{aligned}
& d_j \left( \frac{(f+1)dt_1}{2\eta_{-1}(v_1)t_1(t_1-1)(ft_1+1)} \cdot \frac{t_1(f t_1+1)(t_j-1)}{(f+1)(t_1-t_j)} \phi_{b+1}(t_1) \right) \\
&= d_j \left( \frac{(t_j-1)dt_1}{2\eta_{-1}(v_1)(t_1-1)(t_1-t_j)} \phi_{b+1}(t_1) \right) \\
&= d_j \left( -\frac{\phi_{b+1}(t_1)dt_1}{2\eta_{-1}(v_1)(t_1-1)} + \frac{\phi_{b+1}(t_1)dt_1}{2\eta_{-1}(v_1)(t_1-t_j)} \right) \\
&= \frac{\phi_{b+1}(t_1)B(t_1, t_j)}{2\eta_{-1}(v_1)}, \\
& d_j \left( \frac{(f+1)dt_1}{2\eta_{-1}(v_1)t_1(t_1-1)(ft_1+1)} \cdot \frac{s(t_1)(f s(t_1)+1)(t_j-1)}{(f+1)(s(t_1)-t_j)} \phi_{b+1}(s(t_1)) \right) \\
&= d_j \left( \frac{(f+1)s'(t_1)dt_1}{2\eta_{-1}(v_1)s(t_1)(s(t_1)-1)(f s(t_1)+1)} \cdot \frac{s(t_1)(f s(t_1)+1)(t_j-1)}{(f+1)(s(t_1)-t_j)} \phi_{b+1}(s(t_1)) \right) \\
&= d_j \left( \frac{(t_j-1)s'(t_1)dt_1}{2\eta_{-1}(v_1)(s(t_1)-1)(s(t_1)-t_j)} \phi_{b+1}(s(t_1)) \right) \\
&= d_j \left( -\frac{\phi_{b+1}(s(t_1))s'(t_1)dt_1}{2\eta_{-1}(v_1)(s(t_1)-1)} + \frac{\phi_{b+1}(s(t_1))s'(t_1)dt_1}{2\eta_{-1}(v_1)(s(t_1)-t_j)} \right) \\
&= \frac{\phi_{b+1}(s(t_1))B(s(t_1), t_j)}{2\eta_{-1}(v_1)}.
\end{aligned}$$

We have the following simplified expression

$$\begin{aligned}
& d_2 \cdots d_n \left( \frac{S_2 \cdot (f+1)dt_1}{2\eta_{-1}(v_1)t_1(t_1-1)(ft_1+1)} \right) \\
&= - \sum_{j=2}^n (f(f+1))^{n-2} \sum_{b, b_i, i \in \{2, \dots, n\} \setminus \{j\}} < \tau_b \prod_{k=2, k \neq j}^n \tau_{b_k} \Gamma_g(f) >_g \prod_{k=2, k \neq j}^n d\phi_{b_k}(t_k) \\
& \cdot \left( \frac{\phi_{b+1}(t_1)B(t_1, t_j) + \phi_{b+1}(s(t_1))B(s(t_1), t_j)}{2\eta_{-1}(v_1)} \right).
\end{aligned}$$

To obtain the  $P_n(t_1, t_j)$  term in the BKMP conjecture, we need to switch one of the  $t_1$  with  $s(t_1)$  in the above expression. This can be done by the following estimate:

$$\left( \frac{B(t_1, t_j)}{2\eta_{-1}(v_1)} (\phi_{b+1}(t_1) + \phi_{b+1}(s(t_1))) \right)_+ = \left( -\frac{B(t_1, t_j)F_{b+1}(w_1)}{\eta_{-1}(v_1)} \right)_+ = 0,$$

and similarly

$$\left( \frac{B(s(t_1), t_j)}{2\eta_{-1}(v_1)} (\phi_{b+1}(t_1) + \phi_{b+1}(s(t_1))) \right)_+ = \left( -\frac{B(s(t_1), t_j)F_{b+1}(w_1)}{\eta_{-1}(v_1)} \right)_+ = 0.$$

Thus we have the relation

$$\begin{aligned}
P_b(t_1, t_j) dt_1 dt_j &= \left( \frac{\phi_{b+1}(s(t_1))B(t_1, t_j) + \phi_{b+1}(t_1)B(s(t_1), t_j)}{\log(1 + \frac{1}{ft}) - \log(1 + \frac{1}{fs(t)})} \right)_+ \\
&= \left( \frac{\phi_{b+1}(t_1)B(t_1, t_j) + \phi_{b+1}(s(t_1))B(s(t_1), t_j)}{2\eta_{-1}(v_1)} \right)_+
\end{aligned}$$

and the estimate of  $\pi_* T_4$  and  $S_2$ :

$$\begin{aligned} d_2 \cdots d_n \left( \frac{\pi_* T_4 \cdot (f+1) dt_1}{2\eta_{-1}(v_1)t_1(t_1)(ft_1+1)} \right)_+ &= d_2 \cdots d_n \left( \frac{S_2 \cdot (f+1) dt_1}{2\eta_{-1}(v_1)t_1(t_1)(ft_1+1)} \right)_+ \\ &= - (f(f+1))^{n-2} \sum_{j=2}^n \sum_{b, b_i, i \in \{2, \dots, n\} \setminus \{j\}} < \tau_b \prod_{k=1, k \neq i, j}^n \tau_{b_k} \Gamma_g(f) >_g \cdot P_b(t_1, t_j) \prod_{k=2, k \neq j}^n d\phi_{b_k}(t_k). \end{aligned}$$

Collect all the pieces, we have finish the computation of the push forward of the RHS of the Symmetrized Cut-Join Equation:

$$\begin{aligned} d_2 \cdots d_n \left( \frac{\pi_*(T_1 + T_2 + T_3 + T_4) \cdot (f+1) dt_1}{2\eta_{-1}(v_1)t_1(t_1)(ft_1+1)} \right)_+ \\ = (f(f+1))^n \sum_{a_1, a_2, b_2, \dots, b_n} < \tau_{a_1} \tau_{a_2} \prod_{k=2}^n \Gamma_{g-1}(f) >_{g-1} \prod_{k=2}^n d\phi_{b_k}(t_k) \cdot P_{a_1, a_2}(t_1) dt_1 \\ - (f(f+1))^{n-1} \sum_{\substack{\text{stable} \\ g_1+g+2=g, I \coprod J = \{2, \dots, n\}}} \sum_{a_1, a_2, b_2, \dots, b_n} < \tau_{a_1} \prod_{i \in I} \tau_{b_i} \Gamma_{g_1}(f) >_{g_1} \\ \cdot < \tau_{a_2} \prod_{j \in J} \tau_{b_j} \Gamma_{g_2}(f) >_{g_2} \prod_{k=2}^n d\phi_{b_k}(t_k) \cdot P_{a_1, a_2}(t_1) dt_1 \\ - (f(f+1))^{n-2} \sum_{j=2}^n \sum_{b, b_i, i \in \{2, \dots, n\} \setminus \{j\}} < \tau_b \prod_{k=2, k \neq j}^n \tau_{b_k} > \prod_{k=2, k \neq j}^n d\phi_{b_k}(t_k) \cdot P_b(t_1, t_j). \end{aligned}$$

The recursion in Theorem 4.5 then follows from plugging in the results of this and the previous section into the equality  $L = \pi_*(T_1 + T_2 + T_3 + T_4)$ .

## REFERENCES

- [1] G. Borot, B. Eynard, M. Mulase, B. Safnuk. *A matrix model for simple Hurwitz numbers, and topological recursion*, arXiv:0906.1206 (2009).
- [2] V. Bouchard, A. Klemm, M. Marino, S. Pasquetti. *Remodeling the B-model*, Commun. Math. Phys. **287** (2009), 117-178.
- [3] L. Chen. *Symmetrized Cut-Join Equation of Marino-Vafa Formula*, Pacific. J. Math. **235**, no. 2, (2008), page 201-212.
- [4] B. Eynard, M. Mulase, B. Safnuk. *The Laplace Transform of The Cut-And-Join Equation and The Bouchard-Marino Conjecture on Hurwitz Numbers*, arXiv:0907.5224v2 (2009).
- [5] I.P. Goulden, D. M. Jackson and A. Vainshtein. *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. Combinatorics **4** (2000) 27-46.
- [6] S. Katz, C.-C. Liu. *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc*, Adv.Theor.Math.Phys. **5** (2002), 1-49.
- [7] J. Li, C.-C. Liu, K. Liu, J. Zhou. *A mathematical theory of topological vertex*, Geometry and Topology **13** (2009), 527621.
- [8] C.-C. Liu, K. Liu, J. Zhou. *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65** (2003), 289-340.
- [9] C.-C. Liu, K. Liu, J. Zhou. *A formula of two-partition Hodge integrals*, J. of AMS, **20**, No.1, (2007), 149184.

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